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ON THE COMPUTATION OF THE COEFFICIENTS ASSOCIATED WITH HIGH ORDER NORMAL FORMS

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A new procedure for obtaining *high order normal forms* and the associated coefficients is presented. It is assumed that the Jacobian of the system considered is in a diagonal form. In comparison with existing normal form approaches, this procedure lends itself more readily to symbolic calculations, like MAPLE, and the calculations of *high order normal forms*, together with the associated coefficients, are carried out much more conveniently. To illustrate the approach, five examples are presented. Examples 1 and 3 also contain a comparison of the results obtained by the methods of normal forms and averaging.

1. INTRODUCTION

If the Jacobian matrix of a non-linear differential equation is in a diagonal form, normal forms are composed of resonant monomials [1–3] which are relatively easy to determine. But obtaining the coefficients associated with each term of a normal form may be quite cumbersome and may pose especially serious difficulties. In spite of the efforts, the conventional normal form methods are not very convenient to apply, requiring a great deal of labor, especially for the calculation of the coefficients of *high order normal forms*. A modified normal form approach has been presented in references [4–6] which facilitates the calculation of normal forms and the associated coefficients. The advantages of this approach have been described and demonstrated in reference [4]. However, if *high order normal forms* are required, the approach introduced in reference [4] is not very convenient to apply, even though it represents a major improvement compared to the conventional normal form theory. Some effort has been made to calculate the associated coefficients of "normal forms"—or certain simplified forms—by other methods, like the method of averaging [7, 8] or multiple scale method [8, 9]. Normal form methods determine normal forms by assuming that the transformed equation contains only resonant polynomial [10–18]. Different procedures of analysis may lead to identical results in low order cases, but a general proof that both the normal form approaches and the methods of averaging lead to identical results in all cases is not easy to establish. Indeed, it is demonstrated that these methods may lead to apparently different results in high order cases, (see examples 1 and 3 in this paper). However, both methods may be considered essentially equivalent to

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each other in the sense that the results which appear to be different are, in fact, linked together through a near identity transformation under some conditions. A detailed discussion of the relationship concerning these methods is presented in reference [10].

In this paper, the modified normal form approach presented in reference [4] is further extended and a new procedure is introduced which lends itself readily to symbolic calculations, like MAPLE. A general MAPLE program is also presented in Appendix A. This program is for determining *high order normal forms* and the associated coefficients, and is based on the procedure introduced in this paper. Five examples are presented. It is noted that by the procedure presented here, *high order normal forms* (including the associated coefficients) are obtained in a very short time. Another MAPLE program is also presented in Appendix B, which is used to verify and confirm the results obtained by the procedure presented in this paper.

2. THE MODIFIED NORMAL FORM APPROACH AND PROCEDURE

Consider the non-linear system described by

$$\dot{x} = Ax + \sum_{m=2}^{k} F^m(x), \tag{1}$$

where $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$; $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues; and $\mathrm{Re}(\lambda_j) = 0$ (assumed) for $j = 1, \ldots, n$. Also, $F^m \in H_n^m$, and H_n^m is a vector space of homogeneous polynomials of order m in n variables with values in C^n.

Introducing a sequence of near identity transformations

$$x = y_2 + P^2(y_2), \quad y_{s-1} = y_s + P^s(y_s), \quad s = 3, 4, \ldots, k \tag{2}$$

into equation (1), one has

$$\begin{cases} \dot{y}_2 = Ay_2 + F_1^2(y_2) + \sum_{n=3}^{k} F_1^n(y_2), \\ \dot{y}_3 = Ay_3 + \sum_{n=2}^{3} F_{n-1}^n(y_3) + \sum_{n=4}^{k} F_2^n(y_3) \quad \text{for } s = 3, \\ \quad \cdots \\ \dot{y}_m = Ay_m + \sum_{n=2}^{k} F_{m-1}^n(y_m) \quad \text{for } s = m, \\ \quad \cdots \\ \dot{y}_k = Ay_k + \sum_{n=2}^{k} F_{n-1}^n(y_k) \quad \text{for } s = k, \end{cases} \tag{3}$$

where $P^s(y_s) \in H_n^s(y_s)$ are undefined polynomials to be determined such that the terms of order s in the transformed form will be simplified as resonant polynomials of order s.

Consider equation (3); introducing transformation

$$y_{m-1} = y_m + P^m(y_m) \tag{4}$$

into the $(m-1)$th order transformed equation

$$\dot{y}_{m-1} = Ay_{m-1} + \sum_{n=2}^{k} F^n_{m-2}(y_{m-1}) \tag{5}$$

yields

$$\dot{y}_m = (I + DP^m)^{-1}\left(Ay_m + AP^m(y_m) + \sum_{n=2}^{k} \tilde{F}^n_{m-2}(y_m)\right), \tag{6}$$

where

$$\sum_{n=2}^{k} \tilde{F}^s_{m-2}(y_m) = \sum_{n=2}^{k} F^n_{m-2}(y_m + P^m(y_m)) = \sum_{n=2}^{k}\sum_{s=1}^{n} \frac{D^s F^n_{m-2}(z)}{s!}(P^m(y_m))^s.$$

Consider the mth order transformed equation given by

$$\dot{y}_m = Ay_m + \sum_{n=2}^{k} F^n_{m-1}(y_m). \tag{7}$$

Here, functions F^n_{m-1} are transformed polynomials, which can be calculated from equation (6), and the results for $n \leqslant 5$ are given as follows:

$$F^2_1 = F^2 + AP^2 - DP^2Ay_2, \qquad F^4_2 = F^4_1 + DF^2_1P^3 - DP^3F^2_1,$$

$$F^3_1 = F^3 + DF^2P^2 - DP^2F^2_1, \qquad F^5_2 = F^5_1 + DF^3_1P^3 - DP^3F^3_2,$$

$$F^4_1 = F^4 + \tfrac{1}{2}D^2F^2(P^2)^2 + DF^3P^2 - DP^2F^3_1, \quad \text{and} \quad F^4_3 = F^4_2 + AP^4 - DP^4Ay_4,$$

$$F^5_1 = F^5 + \tfrac{1}{2}D^2F^3(P^2)^2 + DF^4P^2 - DP^2F^4_1, \qquad F^5_3 = F^5_2 + DF^2_1P^4 - DP^4F^2_1,$$

$$F^3_2 = F^3_1 + AP^3 - DP^3Ay_3, \qquad F^5_4 = F^5_3 + AP^5 - DP^5Ay_5.$$

The transformed polynomials $F^n_{m-1}(y_m)$ can be obtained by symbolic calculations readily, using for example, MAPLE.

Let the kth equation in equation (3) be written as

$$\dot{y} = Ay + \sum_{n=2}^{k} F^n_{n-1}(y). \tag{8}$$

Suppose that $F^s_{x-1}(y) = G^s_{NF}(y)$ $(s = 2, 3, \dots, k)$ in equation (8), where $G^s_{NF}(y)$ are the resonant polynomials of order s. Solving $P^2(y)$ from $F^2_1(y, P^2) = G^2_{NF}(y)$, and substituting $P^2(y)$ into $F^3_1(y, P^2)$ define F^3_1 and $F^3_1(y)$. Then, solving $P^3(y)$ from $F^3_2(y, P^3) = G^3_{NF}(y)$, the coefficients in $G^3_{NF}(y)$ can be determined. This procedure is not very convenient and one needs to solve a series of algebraic equations. It becomes increasingly more difficult as the order of the normal form increases [3, 4].

Next, introduce the transformation

$$y = e^{At}z \tag{9}$$

into equations (8) to obtain

$$\dot{z} = \mathrm{e}^{-At} \sum_{m=2}^{k} F_{m-1}^{m}(\mathrm{e}^{At}z), \tag{10}$$

where $\mathrm{e}^{At}z = (\mathrm{e}^{\lambda_1 t}z_1 \;\; \mathrm{e}^{\lambda_2 t}z_2, \dots, \mathrm{e}^{\lambda_n t}z_n)^{\mathrm{T}}$.

It was proved in reference [4] that if $F_{s-1}^{s}(y) = G_{NF}^{s}(y)$, one has

$$\mathrm{e}^{-At}F_{s-1}^{s}(e^{At}z) = F_{s-1}^{s}(z) = G_{NF}^{s}(y) = \underset{t}{M}\{\mathrm{e}^{-At}F_{s-1}^{s}(\mathrm{e}^{At}z)\}, \tag{11}$$

where $M_t\{f(z,t)\}$ denotes explicit time averaging of function $f(z,t)$.

Thus, equation (10) can be expressed as

$$\dot{z} = \sum_{m=2}^{k} \underset{t}{M}\{\mathrm{e}^{-At}F_{m-1}^{m}(e^{At}z)\} = \sum_{m=2}^{k} G_{NT}^{m}(z). \tag{12}$$

It is noted that $\dot{z}$ is not assumed to be the average value of $\dot{z}$ over T—like an averaging method does—but rather, this is a result of the formulation and transformations introduced such that $\dot{z} = f(z,t)$ becomes $\dot{z} = f(z)$. The approach itself is basically not an averaging procedure; rather it uses the underlying ideas of normal form theory.

Carrying out the transformation $z = \mathrm{e}^{-At}x$ into equation (12), one has

$$\dot{x} = Ax + \sum_{m=2}^{k} G_{NT}^{m}(x). \tag{13}$$

This is a normal form of equation (1).

It is evident that the results of this approach are identical to those of the conventional normal form theory. Consider the following relations:

$$\begin{aligned} &\mathrm{e}^{-At}(DP^k(\mathrm{e}^{At}z)A\mathrm{e}^{At}z - AP^k(\mathrm{e}^{At}z)) = \frac{\partial}{\partial t}[\mathrm{e}^{-At}P^k(\mathrm{e}^{At}z)], \\ &F_{k-1}^{k} = F_{k-2}^{k} + AP^k - DP^kAy. \end{aligned} \tag{14}$$

Then, one has

$$\mathrm{e}^{-At}F_{s-1}^{s}(\mathrm{e}^{At}z) - \mathrm{e}^{-At}F_{s-2}^{s}(\mathrm{e}^{At}z) = \frac{\partial}{\partial t}[\mathrm{e}^{-At}P^s(e^{At}z)]. \tag{15}$$

Equations (14) and (15) lead to

$$\left\{\begin{aligned} &G_{NF}^{2} + \frac{\partial}{\partial t}(\mathrm{e}^{-At}P^2(\mathrm{e}^{At}z)) = \mathrm{e}^{-At}F^2(\mathrm{e}^{At}z), \\ &G_{NF}^{3} + \frac{\partial}{\partial t}(\mathrm{e}^{-At}P^3(\mathrm{e}^{At}z)) = \mathrm{e}^{-At}F_1^3(\mathrm{e}^{At}z), \\ &\dots \\ &G_{NF}^{k} + \frac{\partial}{\partial t}(\mathrm{e}^{-At}P^k(\mathrm{e}^{At}z)) = \mathrm{e}^{-At}F_{k-2}^{k}(\mathrm{e}^{At}z). \end{aligned}\right. \tag{16}$$

It is observed from the above analysis that the kth order normal form (with the associated coefficients) can be obtained directly from the $(k-2)$th transformed functions.

It was proved in reference [4] that

$$G_{NF}^{s}(z) = \mathrm{e}^{-At}F_{s-1}^{s}(\mathrm{e}^{At}z) = \underset{t}{M}\{\mathrm{e}^{-At}F_{s-1}^{s}(\mathrm{e}^{At}z)\} = \underset{t}{M}\{\mathrm{e}^{-At}F_{s-2}^{s}(\mathrm{e}^{At})\}. \tag{17}$$

Thus, normal forms of order s are obtained as follows:

$$G_{NF}^{s}(z) = \begin{pmatrix} \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_1=0}} a_{s_1 s_2 \cdots s_n(1)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \\ \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_2=0}} a_{s_1 s_2 \cdots s_n(2)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \\ \cdots \\ \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_n=0}} a_{s_1 s_2 \cdots s_n(n)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \end{pmatrix}, \tag{18}$$

where $\bar{s} = s_1 + s_2 + \cdots + s_n$; $\delta = s_1\lambda_1 + s_2\lambda_2 + \cdots + s_n\lambda_n$; $k \geqslant 2$. $a_{s_1 s_2 \cdots s_n(m)}^{k-2}$ are the coefficients of transformed function $F_{k-2}^{k}(z)$.

Equation (16) leads to

$$\begin{aligned} P^{s}(z) &= \mathrm{e}^{-At}P^{s}(\mathrm{e}^{At}z)|_{t=0} \\ &= \int \mathrm{e}^{-At}(F_{s-2}^{s}(e^{At}z) - F_{s-1}^{s}(\mathrm{e}^{At}z))\,\mathrm{d}t\,|_{t=0} = \int [\mathrm{e}^{-At}F_{s-2}^{s}(\mathrm{e}^{At}z) - G_{NF}^{s}(z)]\,\mathrm{d}t\,|_{t=0}. \end{aligned} \tag{19}$$

Solving the above equation for $P^s(z)$, $P^s(z)$ can be obtained readily as follows:

$$P^{s}(z) = \begin{pmatrix} \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_1\neq 0}} \frac{1}{\delta-\lambda_1} a_{s_1 s_2 \cdots s_n(1)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \\ \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_2\neq 0}} \frac{1}{\delta-\lambda_2} a_{s_1 s_2 \cdots s_n(2)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \\ \cdots \\ \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_n\neq 0}} \frac{1}{\delta-\lambda_n} a_{s_1 s_2 \cdots s_n(n)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \end{pmatrix} + \begin{pmatrix} \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_1=0}} c_{s_1 s_2 \cdots s_n(1)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \\ \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_2=0}} c_{s_1 s_2 \cdots s_n(2)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \\ \cdots \\ \displaystyle\sum_{\substack{\bar{s}=s \\ \delta-\lambda_n=0}} c_{s_1 s_2 \cdots s_n(n)}^{s-2} z_1^{s_1} z_2^{s_2} \cdots z_n^{s_n} \end{pmatrix}, \tag{20}$$

where for convenience, $c_{s_1 s_2 \cdots s_n(k)}^{s-2}$, $k = 1, \ldots, n$, are usually chosen as zero.

Normal forms and the transformation functions can be determined from equations (16), (18) and (20). It is clear that one does not need to solve any differential or algebraic equations. Algebraic calculations are sufficient and only simple co-ordinate transformations are involved in the above procedure. It presents a major improvement compared to the conventional normal form theory. The examples presented in reference [4] have clearly demonstrated that the above normal form approach provides a number of significant advantages over the existing normal from theory which involves inverting a set of complex matrices (see equation (6)). This may take a long time and occupy a lot of memory in MAPLE calculations, especially in the calculations of *high order normal forms*, and it becomes increasingly more inconvenient as the order of the normal form increases. In reference [4], the explicit expressions of the transformation functions F_m^n are considered (in MAPLE calculations) which does not require the calculation of the inverses of a set of matrices. However, the calculations are not based on an iterative procedure. Therefore if *high order normal forms* are required, the approach introduced in reference [4] is not very convenient to apply. In order to render the above procedure even more applicable (with regard to symbolic calculations), consider the identity

$$(I + DP^m)^{-1} = I - DP^m(I + DP^m)^{-1}. \tag{21}$$

Introducing equation (21) into equation (6) leads to

$$\dot{y}_m = Ay_m + AP^m(y_m) + \sum_{n=2}^{k} \tilde{F}^n_{m-2}(y_m) - DP^m\left(Ay_m + \sum_{n=2}^{k} F^n_{m-1}(y_m)\right). \tag{22}$$

Thus, in general, one has

$$\begin{aligned} F^s_{m-1} &= F^s_{m-2}, && s < m, \\ F^s_{m-1} &= F^s_{m-2} + AP^m - DP^m A y_m, && s = m, \\ F^s_{m-1} &= \tilde{F}^s_{m-2} - DP^m F^{s-m+1}_{m-1}, && s > m. \end{aligned} \tag{23}$$

It is noted that the above formulation introduces an iteration procedure and does not include any calculation of the inverse of a matrix. It is convenient to determine normal forms and the associated coefficients from such an iterative procedure.

It is clear that the second iteration equation in equation (23) produces the normal form of order m. According to equations (18) and (20), the normal form of order m, F^m_{m-1}, and the $(m+1)$th order transformation function P^{m+1} are given by

$$F^m_{m-1} = \begin{pmatrix} \displaystyle\sum_{\substack{\bar{s}=m \\ \delta-\lambda_1=0}} a^{m-2}_{s_1 s_2 \cdots s_n(1)} y^{s_1}_{m1} y^{s_2}_{m2} \cdots y^{s_n}_{mn} \\ \displaystyle\sum_{\substack{\bar{s}=m \\ \delta-\lambda_2=0}} a^{m-2}_{s_1 s_2 \cdots s_n(2)} y^{s_1}_{m1} y^{s_2}_{m2} \cdots y^{s_n}_{mn} \\ \cdots \\ \displaystyle\sum_{\substack{\bar{s}=m \\ \delta-\lambda_n=0}} a^{m-2}_{s_1 s_2 \cdots s_n(n)} y^{s_1}_{m1} y^{s_2}_{m2} \cdots y^{s_n}_{mn} \end{pmatrix} \quad \text{and}$$

$$P^{m+1}(z) = \begin{pmatrix} \displaystyle\sum_{\substack{\bar{s}=m+1 \\ \delta-\lambda_1\neq 0}} \frac{1}{\delta-\lambda_1} a^{m-1}_{s_1 s_2 \cdots s_n(1)} y^{s_1}_{(m+1)1} \cdots y^{s_n}_{(m+1)n} \\ \displaystyle\sum_{\substack{\bar{s}=m+1 \\ \delta-\lambda_2\neq 0}} \frac{1}{\delta-\lambda_2} a^{m-1}_{s_1 s_2 \cdots s_n(2)} y^{s_1}_{(m+1)1} \cdots y^{s_n}_{(m+1)n} \\ \cdots \\ \displaystyle\sum_{\substack{\bar{s}=m+1 \\ \delta-\lambda_n\neq 0}} \frac{1}{\delta-\lambda_n} a^{m-1}_{s_1 s_2 \cdots s_n(n)} y^{s_1}_{(m+1)1} \cdots y^{s_n}_{(m+1)n} \end{pmatrix}.$$

Here $a^{m-1}_{s_1 s_2 \cdots s_n(k)}$ are the coefficients of transformed function $F^{m+1}_{m-1}(y)$, which has been determined from equation (23).

Then, the iteration procedure of equation (23) can be completed to any order k, which starts with

$$F_1^2 = F^2 + AP^2 - DP^2 A y_2, \quad F_1^s = \tilde{F}^2 - DP^2 F_1^{s-1},\ s > 2, \quad \text{and} \quad P^2(z) = \begin{pmatrix} \displaystyle\sum_{\substack{\bar{s}=2 \\ \delta-\lambda_1 \neq 0}} \frac{1}{\delta - \lambda_1} a_{s_1 s_2 \cdots s_n(1)} y_{21}^{s_1} \cdots y_{2n}^{s_n} \\ \displaystyle\sum_{\substack{\bar{s}=2 \\ \delta-\lambda_2 \neq 0}} \frac{1}{\delta - \lambda_2} a_{s_1 s_2 \cdots s_n(2)} y_{21}^{s_1} \cdots y_{2n}^{s_n} \\ \cdots \\ \displaystyle\sum_{\substack{\bar{s}=2 \\ \delta-\lambda_n \neq 0}} \frac{1}{\delta - \lambda_n} a_{s_1 s_2 \cdots s_n(n)} y_{21}^{s_1} \cdots y_{2n}^{s_n} \end{pmatrix}.$$

Normal forms (and the associated coefficients) can be determined from the above procedure, as in equation (23). It is clear that equation (23) involves a simple iteration procedure only and it is very convenient to apply a symbolic calculation program. For instance, a MAPLE program is designed in Appendix A, which is based on equation (23), to determine the normal form and the associated coefficients of two-dimensional systems. Based on equation (23), it is very convenient to extend the program to determine the normal form and the associated coefficients of higher order systems.

In the above iteration procedure, in order to determine the normal form of order m, G_{NF}^m $(= F_{m-1}^m)$, and the $(m+1)$th transformation function P^{m+1}, one needs to determine all the terms of order m which satisfy the condition $\delta - \lambda_k = 0$ in equation (18), and to determine all the terms of order $m+1$ which satisfy the condition $\delta - \lambda_k \neq 0$ in equation (20). For example, consider a polynomial described by two variables, given by

$$g(y) = \sum a_{s_1 s_2} y_1^{s_1} y_2^{s_2}.$$

One needs to determine the terms of order m and satisfy the condition $\delta - \lambda_k \equiv s_1(\omega i) + s_2'(-\omega i) - \lambda_k = 0$, where $\lambda_k = \pm\omega i$. This can be completed by introducing $y_1 = suz_1$, $y_2 = su^{-1}z_2$ into the above polynomial, which results in

$$g_k(z) = \sum s^{(s_1+s_2)} u^{(s_1-s_2)} a_{s_1 s_2(k)} z_1^{s_1} z_2^{s_2}, \quad k = 1, 2,$$

where s, u are constants.

Thus, the terms of order m which satisfy the condition $s_1(\omega i) + s_2\,(-\omega i) = \lambda_k$ can be determined as the coefficients of $s^m u^{\pm 1}$. It is obtained readily by a simple procedure in the MAPLE program.

Another MAPLE program is presented to verify and confirm (with regard to the conventional normal form procedure) the results obtained by the above procedure and program. The transformation functions P^s, obtained from the above program, are sequentially put back into the original equation (1), and the transformed functions F_{m-1}^m are determined as in equation (6), which is based on the conventional normal form procedure. If the transformed functions F_{m-1}^m consist of terms which satisfy the resonant condition

$$s_1\lambda_1 + s_2\lambda_2 + \cdots + s_n\lambda_n - \lambda_s = 0, \quad s = 1, 2, \ldots, n,$$

then F_{m-1}^m are normal forms. This is the basic procedure of the conventional normal form theory. If normal forms F_{m-1}^m are identical to those obtained from program 1, the results by program 1 are verified and confirmed as the right ones, in the sense of the conventional normal form theory.

Next, five examples are presented to demonstrate the convenience of this approach and procedure.

3. APPLICATIONS

Example 1. Determine the normal form of the two-dimensional system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + ax^2 y. \end{aligned} \tag{24}$$

First, transforming equation (24) into complex form by using

$$\begin{aligned} x &= \tfrac{1}{2}(z_1 + z_2), \\ y &= \tfrac{1}{2} i(z_1 - z_2) \end{aligned}$$

yields

$$\dot{z} = Az + F^3(z), \tag{25}$$

where $\mathrm{A} = diag(i, -i)$; $z = (z_1 \ \ z_2)^{\mathrm{T}}$ and $z_2 = \bar{z}_1$.

According to the discussion in the previous sections, a normal form (up to order 11) of equation (25) is in the following form:

$$\dot{z} = Az + G^3_{NT}(z) + G^5_{NT}(z) + G^7_{NT}(z) + G^9_{NT}(z) + G^{11}_{NT}(z).$$

Transforming to polar co-ordinates leads to

$$\begin{aligned} \dot{r} &= a_1 r^3 + a_2 r^5 + a_3 r^7 + a_4 r^9 + a_5 r^{11}, \\ \dot{\theta} &= \omega + b_1 r^2 + b_2 r^4 + b_3 r^6 + b_4 r^8 + b_5 r^{10}. \end{aligned} \tag{26}$$

Thus, one has the normal form up to order 11 (obtained by the program in Appendix A) as follows:

$$\begin{aligned} \dot{z}_1 = {} & \mathrm{i} z_1 + \tfrac{1}{8} a z_1^2 z_2 - \tfrac{11}{256} \mathrm{i} a^2 z_1^3 z_2^2 + \tfrac{13}{8192} a^3 z_1^4 z_1^3 - \tfrac{1321}{786\,432} \mathrm{i} a^4 z_1^5 z_2^4 \\ & + \tfrac{7727}{18\,874\,368} a^5 z_1^6 z_2^5 + o(r^{12}), \\ \dot{z}_2 = {} & -\mathrm{i} z_2 + \tfrac{1}{8} a z_2^2 z_1 + \tfrac{11}{256} \mathrm{i} a^2 z_2^3 z_1^2 + \tfrac{13}{8192} a^3 z_2^4 z_1^3 + \tfrac{1321}{786\,432} \mathrm{i} a^4 z_2^5 z_1^4 \\ & + \tfrac{7727}{18\,874\,368} a^5 z_2^6 z_1^5 + o(r^{12}). \end{aligned} \tag{27}$$

In polar co-ordinates, one has

$$\begin{aligned} \dot{r} &= \tfrac{1}{8} a r^3 + \tfrac{13}{8192} a^3 r^7 + \tfrac{7727}{18\,874\,368} a^5 r^{11} + o(r^{12}), \\ \dot{\theta} &= 1 - \tfrac{11}{256} a^2 r^4 - \tfrac{1321}{786\,432} a^4 r^8 + o(r^{12}). \end{aligned} \tag{28}$$

The calculation time by PC (CPU 266) computer is very short (about 2·5 s).

Consider the transformation functions P^n $(n = 2, 3, \ldots, 11)$, obtained from program 1. Introducing the transformation $z = x + P^n(x)$ sequentially into the original equation (25) and following the original procedure of the conventional normal form theory as in equation

(6) produces results identical to those given in equation (27). This verifies and confirms the results obtained by program 1.

In order to compare the above results with those by an averaging method, equation (25) is considered. Introducing the scaling $z = \varepsilon z$ into equation (25) results in

$$\dot{z} = Az + \varepsilon^2 F^3(z), \tag{29}$$

where ε is a small perturbation parameter.

Substituting $z = e^{At}y$ into equation (29), one has

$$\dot{y} = \varepsilon^2 f^3(y, t), \tag{30}$$

in which $f^3(y, t) = e^{-At}F^3(e^{At}y)$.

Substituting transformations

$$y = x + \sum_{m=1}^{s} \varepsilon^m \phi_m(x, t) \tag{31}$$

and

$$\dot{x} = \sum_{m=1}^{s} \varepsilon^m G_A^m(x) \tag{32}$$

into equation (30) results in

$$\left(I + \sum_{m=1}^{s} \varepsilon^m \frac{\partial \phi_m}{\partial x}\right)\left(\sum_{m=1}^{s} \varepsilon^m G_A^m\right) + \sum_{m=1}^{s} \varepsilon^m \frac{\partial \phi_m}{\partial t} = \varepsilon^2 f^3\left(x + \sum_{i=1}^{s} \varepsilon^i \phi_i, t\right). \tag{33}$$

Comparing the coefficients of similarly ordered terms in equation (33) produces

$$\varepsilon\colon\ G_A^1 + \frac{\partial \phi_1}{\partial t} = f_A^2,$$

$$\varepsilon^2\colon\ G_A^2 + \frac{\partial \phi_2}{\partial t} = f_A^3,$$

$$\cdots$$

$$\varepsilon^{k-1}\colon\ G_A^{k-1} + \frac{\partial \phi_{k-1}}{\partial t} = f_A^k, \tag{34}$$

where

$$f_A^2 = f^2 = 0,$$

$$f_A^3 = f^3 + D_x f^2 \phi_1 - D_x \phi_1 G_A^1,$$

$$f_A^4 = D_x f^2 \phi_2 + D_x f^3 \phi_1 + \tfrac{1}{2} D_x^2 f^2 \phi_1^2 - D_x \phi_2 G_A^1 - D_x \phi_1 G_A^2,$$

$$\cdots$$

$$f_A^k = \sum_{m=2}^{k} \sum_{p=1}^{m} \frac{D_x^p f^m}{p!} \sum_{\substack{\bar{s}=p \\ \delta=k-m}} a_{s_1 s_2 \cdots s_s} \phi_1^{s_1} \phi_2^{s_2} \cdots \phi_s^{s_s} - \sum_{m=1}^{k-1} D_x \phi_{k-m-1} G_A^m$$

and

$$\phi_q = \int (f_A^{q+1} - G_A^q)\,\mathrm{d}t + \sum_{r=1}^{k} \begin{pmatrix} \tilde{c}_{r(1)} G_{A(1)}^r \\ \cdots \\ \tilde{c}_{r(n)} G_{A(n)}^r \end{pmatrix}, \quad G_A^r = \begin{pmatrix} G_{A(1)}^r(x) \\ \cdots \\ G_{A(n)}^r(x) \end{pmatrix}.$$

For convenience, $\tilde{c}_{r(1)}, \tilde{c}_{r(2)}, \ldots, \tilde{c}_{r(n)}$ are usually chosen as zero.

Thus, the results by the above averaging method are given by

$$\begin{aligned} \dot{x}_1 &= \mathrm{i}x_1 + \tfrac{1}{8} a x_1^2 x_2 - \tfrac{11}{256}\mathrm{i}a^2 x_1^3 x_2^2 + \tfrac{13}{8192} a^3 x_1^4 x_2^3 - \tfrac{3319}{786\,432}\mathrm{i}a^4 x_1^5 x_2^4 \\ &\quad + \tfrac{2035}{37\,748\,736} a^5 x_1^6 x_2^5 + o(r^{12}), \\ \dot{x}_2 &= -\mathrm{i}x_2 + \tfrac{1}{8} a x_2^2 x_1 + \tfrac{11}{256}\mathrm{i}a^2 x_2^3 x_1^2 + \tfrac{13}{8192} a^3 x_2^4 x_1^3 + \tfrac{3319}{786\,432}\mathrm{i}a^4 x_2^5 x_1^4 \\ &\quad + \tfrac{2035}{37\,748\,736} a^5 x_2^6 x_1^5 + o(r^{12}). \end{aligned} \tag{35}$$

In polar co-ordinates, one has

$$\begin{aligned} \dot{r} &= \tfrac{1}{8} a r^3 + \tfrac{13}{8192} a^3 r^7 + \tfrac{2035}{37\,748\,736} a^5 r^{11} + o(r^{12}), \\ \dot{\theta} &= 1 - \tfrac{11}{256} a^2 r^4 - \tfrac{3319}{786\,432} a^4 r^8 + o(r^{12}). \end{aligned} \tag{36}$$

It is interesting to note that the results obtained by both methods are identical (in this example) up to order seven, and appear to be different at higher orders. However, it can be shown that the complete results (including terms of order 11) are linked together by a near identity transformation:

$$\begin{aligned} x_1 &= z_1 + \tfrac{333}{16\,384}\mathrm{i}a^3 z_1^4 z_2^3 - \tfrac{181}{131\,072} a^4 z_1^5 z_2^4, \\ x_2 &= z_2 - \tfrac{333}{16\,384}\mathrm{i}a^3 z_2^4 z_1^3 - \tfrac{181}{131\,072} a^4 z_2^5 z_1^4. \end{aligned} \tag{37}$$

Introducing the transformation (37) into equation (35) leads to equation (27). One can obtain similar conclusions in all the following examples. A detailed discussion concerning the relationship and differences between the methods of normal forms and averaging is presented in reference [10].

Example 2. Determine the normal form and the related coefficients of the Duffing equation,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \alpha x^3. \end{aligned} \tag{38}$$

Transforming equation (38) into complex form yields

$$\dot{z} = -\mathrm{i}z + \frac{\mathrm{i}\alpha}{8}(z + \bar{z})^3. \tag{39}$$

Following the analysis above, normal forms of equation (39), up to order 11, can be expressed as

$$\dot{z} = Az + G_{NT}^3(z) + G_{NT}^5(z) + G_{NT}^7(z) + G_{NT}^9(z) + G_{NT}^{11}(z). \tag{40}$$

Transforming equation (40) to polar co-ordinates, one has

$$\dot{r} = a_1 r^3 + a_2 r^5 + a_3 r^7 + a_4 r^9 + a_5 r^{11},$$

$$\dot{\theta} = \omega + b_1 r^2 + b_2 r^4 + b_3 r^6 + b_4 r^8 + b_5 r^{10}.$$

After simple iterations, the coefficients of the above equation are obtained as follows:

$$\dot{r} = o(r^{12}),$$

$$\dot{\theta} = \omega - \tfrac{3}{8} a r^2 - \tfrac{51}{256} a^2 r^4 - \tfrac{1419}{8192} a^3 r^6 - \tfrac{50\,691}{262\,144} a^4 r^8 - \tfrac{964\,509}{4\,194\,304} a^5 r^{10}. \tag{41}$$

Using MAPLE the coefficients in equation (41) are obtained in 2·5 s by PC (CPU 266) computer. Verifying by program 2 shows that the results in equation (41) are correct. The normal form up to order 4 agrees with those in reference [11] (The maximum order of normal form is 4 in reference [11]), which are obtained by conventional normal form theory.

Example 3. Determine the normal form and the related coefficients of the following system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + ax^2 + bx^2 y. \end{aligned} \tag{42}$$

Transforming equation (42) into complex form yields

$$\dot{z} = Az + F^2(z) + F^3(z), \tag{42*}$$

where $A = diag(i, -i)$; $z = (z_1 \;\; z_2)^{\mathrm{T}}$ and $z_2 = \bar{z}_1$.

Applying a similar analysis as above, one can obtain the normal form of the above equation, up to order 7, as

$$\begin{aligned} \dot{z}_1 &= \mathrm{i}z_1 + \left(\frac{b}{8} - \frac{5\mathrm{i}a^2}{12}\right) z_1^2 z_2 + \left(\frac{5a^2 b}{72} - \frac{11\mathrm{i}b^2}{256} - \frac{785\mathrm{i}a^4}{1728}\right) z_1^3 z_2^2 \\ &\quad + \left(\frac{13b^3}{8192} + \frac{6167a^4 b}{55\,296} - \frac{4971\mathrm{i}a^2 b^2}{20\,480} - \frac{155\,525\mathrm{i}a^6}{248\,832}\right) z_1^4 z_2^3, \\ \dot{z}_2 &= -\mathrm{i}z_2 + \left(\frac{b}{8} + \frac{5\mathrm{i}a^2}{12}\right) z_2^2 z_1 + \left(\frac{5a^2 b}{72} + \frac{11\mathrm{i}b^2}{256} + \frac{785\mathrm{i}a^4}{1728}\right) z_2^3 z_1^2 \\ &\quad + \left(\frac{13b^3}{8192} + \frac{6167a^4 b}{55\,296} + \frac{4971\mathrm{i}a^2 b^2}{20\,480} + \frac{155\,525\mathrm{i}a^6}{248\,832}\right) z_2^4 z_1^3. \end{aligned} \tag{43}$$

In polar co-ordinates, one has

$$\begin{aligned} \dot{r} &= \frac{b}{8} r^3 + \frac{5a^2 b}{72} r^5 + \left(\frac{13b^3}{8192} + \frac{6167a^4 b}{55\,296}\right) r^7 + o(r^9), \\ \dot{\theta} &= 1 - \frac{5a^2}{12} r^2 - \left(\frac{11b^2}{256} + \frac{785a^4}{1728}\right) r^4 - \left(\frac{4971a^2 b^2}{20\,480} + \frac{155\,525a^6}{248\,832}\right) r^6 + o(r^8). \end{aligned} \tag{44}$$

Using MAPLE, the coefficients in equation (43) are obtained in 9 s by PC (CPU 266) computer. Verifying by program 2 shows that the results (equation (43)) are correct.

In order to compare with an averaging method, one considers

$$\dot{x} = Ax + \varepsilon F^2(x) + \varepsilon^2 F^3(x) \tag{45}$$

upon introducing $z_1 = \varepsilon x_1$, $z_2 = \varepsilon x_2$ into (42*) ($x = (x_1 \;\; x_2)^{\mathrm{T}}$).

Following the averaging analysis as in example 1, one has

$$\dot{x}_1 = \mathrm{i}x_1 + \left(\frac{b}{8} - \frac{5\mathrm{i}a^2}{12}\right)x_1^2 x_2 + \left(\frac{5a^2 b}{72} - \frac{11\mathrm{i}b^2}{256} - \frac{785\mathrm{i}a^4}{1728}\right)x_1^3 x_2^2$$

$$+ \left(\frac{13b^3}{8192} + \frac{10\,309a^4 b}{165\,888} - \frac{39\,619\mathrm{i}a^2 b^2}{184\,320} - \frac{65\,495\mathrm{i}a^6}{82\,944}\right)x_1^4 x_2^3,$$

$$\dot{x}_2 = -\mathrm{i}x_2 + \left(\frac{b}{8} + \frac{5\mathrm{i}a^2}{12}\right)x_2^2 x_1 + \left(\frac{5a^2 b}{72} + \frac{11\mathrm{i}b^2}{256} + \frac{785\mathrm{i}a^4}{1728}\right)x_2^3 x_1^2,$$

$$+ \left(\frac{13b^3}{8192} + \frac{10\,309a^4 b}{165\,888} + \frac{39\,619\mathrm{i}a^2 b^2}{184\,320} + \frac{65\,495\mathrm{i}a^6}{82\,944}\right)x_2^4 x_1^3. \tag{46}$$

In polar co-ordinates, one has

$$\dot{r} = \frac{b}{8}r^3 + \frac{5a^2 b}{72}r^5 + \left(\frac{13b^3}{8192} + \frac{10\,309a^4 b}{165\,888}\right)r^7 + o(r^9),$$

$$\dot{\theta} = 1 - \frac{5a^2}{12}r^2 - \left(\frac{11b^2}{256} + \frac{785a^4}{1728}\right)r^4 - \left(\frac{39\,619a^2 b^2}{184\,320} + \frac{65\,495a^6}{82\,944}\right)r^6 + o(r^8). \tag{47}$$

Comparing the above results with those of the normal form approach, it is found that the results obtained by both methods are identical (in this example) up to order five, and appear to be different at higher orders. However, it can be shown that the complete results (including terms of order seven) are linked together by a near identity transformation

$$x_1 = z_1 - \frac{2a^2(-414a^2b^2 + 243\mathrm{i}b^3 - 2880\mathrm{i}a^4 b + 1600a^6)}{243(9b^2 + 100a^4)}z_1^3 z_2^2,$$

$$x_2 = z_2 - \frac{2a^2(-414a^2b^2 - 243\mathrm{i}b^3 + 2880\mathrm{i}a^4 b + 1600a^6)}{243(9b^2 + 100a^4)}z_2^3 z_1^2, \tag{48}$$

assuming that $9b^2 + 100a^4 \neq 0$. In the case $9b^2 + 100a^4 = 0$, one can find other near identity transformations to link results (43) and (46).

It is important to note that this conclusion has been reached by introducing a scaling of state variables (as $z_1 = \varepsilon x_1$, $z_2 = \varepsilon x_2$). However, many non-linear methods are based on the assumption that the non-linear part of the governing equations is sufficiently small. In other words, in the case of the system under consideration, equation (42*) may be written as

$$\dot{x} = Ax + \varepsilon F^2(x) + \varepsilon F^3(x). \tag{49}$$

Applying a similar averaging technique to this system leads to

$$\dot{r} = \frac{b}{8}r^3 + \frac{89a^2b}{1152}r^5 + \left(\frac{5b^3}{4096} + \frac{6209a^4b}{82\,944}\right)r^7 + o(r^9),$$

$$\dot{\theta} = 1 - \frac{5a^2}{12}r^2 - \left(\frac{b^2}{32} + \frac{785a^4}{1728}\right)r^4 - \left(\frac{7957a^2b^2}{36\,864} + \frac{65\,495a^6}{82\,944}\right)r^6 + o(r^8). \quad (50)$$

It is clear that now equations (50) and (44) are identical only up to order 3, and one can only find a near identity transformation connecting equation (50) to equation (44) under the condition $9b^2 + 100a^4 \neq 0$. In the case of $9b^2 + 100a^4 = 0$ it has been *impossible* to find a near identity transformation linking these results.

Example 4. Determine the normal form and related coefficients of the two-dimensional system with six parameters, given by

$$\begin{aligned} \dot{x} &= -y + \lambda_1 x - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} &= x + \lambda_1 y + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{aligned} \quad (51)$$

Transforming the above equation into a complex co-ordinate form, one has

$$\dot{z} = Az + F^2(z), \quad (52)$$

where

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad F^2(x) = \begin{pmatrix} a_{20}z_1^2 + a_{11}z_1z_2 + a_{02}z_2^2 \\ b_{20}z_1^2 + b_{11}z_1z_2 + b_{02}z_2^2 \end{pmatrix}, \quad a_{mn} = a_{mn}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6).$$

Transforming equation (52) to polar co-ordinates, one has

$$\begin{aligned} \dot{r} &= a_1 r^3 + a_2 r^5 + a_3 r^7, \\ \dot{\theta} &= 1 + O(|r|^2). \end{aligned} \quad (53)$$

After simple iterations, the coefficients of the above equation are obtained as follows:

$$\begin{aligned} a_1 &= -\tfrac{1}{8}\lambda_5(\lambda_3 - \lambda_6) && \text{for } \lambda_1 = 0, \\ a_2 &= \tfrac{1}{48}\lambda_2\lambda_4(\lambda_3 - \lambda_6)[\lambda_4 + 5(\lambda_3 - \lambda_6)] && \text{for } \lambda_1 = \lambda_5 = 0, \\ a_3 &= \tfrac{25}{64}\lambda_2(\lambda_3 - \lambda_6)^3(\lambda_3\lambda_6 - \lambda_2^2 - 2\lambda_6^2) && \text{for } \lambda_1 = \lambda_5 = 0,\ \lambda_4 = -5(\lambda_3 - \lambda_6). \end{aligned} \quad (54)$$

They are identical to the results of Farr *et al.* [12], who employed L-S theory. However, the results here follow a simple procedure and they are obtained in a very fast manner by using the MAPLE program given in Appendix A. Thus, the coefficients in equation (53) are obtained within 2 s by PC (CPU 266) computer.

Example 5. Determine the normal form and the related coefficients of the system given by

$$\dot{x} = y, \qquad \dot{y} = -x + bx^2 + cy^3. \tag{55}$$

Similar to the analysis above, one can obtain the normal form of the above equation, up to order 7, as

$$\dot{z}_1 = \mathrm{i}z_1 + \left(\frac{3c}{8} - \frac{5\mathrm{i}b^2}{12}\right) z_1^2 z_2 - \left(\frac{785\mathrm{i}b^4}{1728} + \frac{5b^2c}{48} - \frac{27\mathrm{i}c^2}{256}\right) z_1^3 z_2^2$$
$$+ \left(\frac{567c^3}{8192} - \frac{113b^4c}{18\,432} - \frac{155\,525\mathrm{i}b^6}{248\,832} - \frac{6739\mathrm{i}c^2b^2}{20\,480}\right) z_1^4 z_2^3,$$
$$\dot{z}_2 = -\mathrm{i}z_2 + \left(\frac{3c}{8} + \frac{5\mathrm{i}b^2}{12}\right) z_2^2 z_1 - \left(-\frac{785\mathrm{i}b^4}{1728} + \frac{5b^2c}{48} - \frac{27\mathrm{i}c^2}{256}\right) z_2^3 z_1^2$$
$$+ \left(\frac{567c^3}{8192} - \frac{113b^4c}{18\,432} + \frac{155\,525\mathrm{i}b^6}{248\,832} + \frac{6739\mathrm{i}c^2b^2}{20\,480}\right) z_2^4 z_1^3. \tag{56}$$

In polar co-ordinates, one has

$$\dot{r} = \frac{3c}{8} r^3 - \frac{5b^2c}{48} r^5 + \left(\frac{567c^3}{8192} - \frac{113b^4c}{18\,432}\right) r^7,$$
$$\dot{\theta} = 1 - \frac{5b^2}{12} r^2 - \left(\frac{27c^2}{256} + \frac{785b^4}{1728}\right) r^4 - \left(\frac{155\,525b^6}{248\,832} + \frac{6739c^2b^2}{20\,480}\right) r^6. \tag{57}$$

Using the MAPLE program, the coefficients in equation (57) are obtained in 9 s by PC (CPU 266) computer. Program 2 shows that the results (equation (57)) are correct.

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APPENDIX A

```
writeto(out);
#order of notmal form
m:=10:w:=1:
gx1:=s*0;gx2:=s^2*(a*u1^3);
u1:=(x1+x2)/2:u2:=I*w*(x1-x2)/2:
fx1:=gx1-1*gx2/w:fx2:=gx1+1*gx2/w:
x1:=e*z1:x2:=e^(-1)*z2:
F1[0]:=expand(e^(-1)*fx1):F2[0]:=expand(e*fx2):
FUN1:=F1[0]:FUN2:=F2[0];
for i from 1 to m do
Fm1[i-1]:=taylor(F1[i-1],s=0,m+1):Fm2[i-1]:=taylor(F2[i-1],s=0,m+1):
pb1[i]:=expand(coeff(Fm1[i-1],s,i)*s^i,e):pb2[i]:=expand(coeff(F:=2[i-1],s,i)*s^i,e):
P1[i]:=0:P2[i] :=0:
for ii from -i-2 to -1 do
P1[i]:= P1[i]+coeff(pb1[i],e,ii)*e,^ii/(ii*I*w):P2[i]:= +coeff(pb2[i],e,ii)*e^ii/(ii*I*w):
od:
for ii from 1 to i+2 do
P1[i]:=P1[i]+coeff(pb1[i],e,ii)*e^ii/(ii*I*w):P2[i]:= P2[i]+coeff(pb2[i],e,ii)*e^ii/(ii*I*w):
od:
dp11[i]:=diff(P1[i],z1):dp12[i]:=diff(P1[i],z2):
dp21[i]:=diff(P2[i],z1):dp22[i]:=diff(P2[i],z2):
F1[i]:= 0:F2[i]:= 0:
if i > 1 then
for k from 1 to i-1 do
FF1[k,i]:= FF1[k,i-1]:FF2[k,i]:=FF2[k,i-1]:
F1[i]:=F1[i]+FF1[k,i]:F2[i]:=F2[i]+FF2[k,i]:
od:fi:
FF1[i,i]:=coeff(pb1[i],e,0):FF2[i,i]:=coeff(pb2[i],e,0):
F1[i]:=F1[i]+FF1[i,i]:F2[i]:= F2[i]FF2[i,i]:
```

```
S1:=subs({z1=z1+P1[i],z2 = z2+P2[i]},F1[i − 1]):S2:= subs({z1=z1+P1[i],z2=z2+P2[i]},F2[
i-1]):
G1:=taylor(S1,s=0,m+1):G2:= taylor(S2,s=0,m+1):
for k from i+1 to m do
FG1[k,i]:= coeff(G1,s,k)*s^(k):FG2[k,i]:= coeff(G2,s,k)*s^(k):
FF1[k,i]:= FG1[k,i]-dp11[i]*FF1[k-i,i]-dp12[i]*FF2[k-i,i]:
FF2[k,i]:= FG2[k,i] − dp21[i]*FF1[k-i,i]-dp22[i]*FF2[k-i,i]:
F1[i]:= F1[i]+FF1[k,i]:F2[i]:= F2[i]+FF2[k,i]:
od:
print(NF1 = ',FF1[i,i]);
#print(NF2 = ',FF2[i,i]);#print('P1',P1[i]);#print('P2',P2[i]);
od:
PP1:= 0; PP2:= 0:
for i from 1 to m do
PP1:=PP1+P1[i]: PP2:=PP+P2[i]:
od:
save PP1, PP2, FUN1, FUN2, 'nfp':
quit:
```

APPENDIX B

```
readlib(mtaylor):
with(linalg):writeto(out):
read nfp:
m:=8;e:=1:w:=1:
aa1[0]:=FUN1:aa2[0]:= FUN2:
for i from 1 to m do
P1[i]:=coeff(PP1,s,i)*s^i:P2[i]:= coeff(PP2,s,i)*s^i:
dp11:=diff(z1+P1[i],z1):dp12:= diff(z1+P1[i],z2):
dp21:=diff(z2+P2[i],z1):dp22:= diff(z2+P2[i],z2):
AP:=array([[dp11,dp12),[dp21,dp22]])):APF:= inverse(AP):
sa11[i]:=mtaylor(APF[1,1],s = 0,m+1):sa12[i]:= mtaylor(APF[1,2],s = 0,m+1):
sa21[i]:=mtaylor(APF[2,1],s = 0,m+1):sa22[i]:= mtaylor(APF[2,2],s = 0,m+1):
AA1[i]:=subs({z1 = z1+P1[i],z2=z2+P2[i]},aa1[i-1);
AA2[i]:=subs({z1 = z1+P1[i],z2=z2+P2[i]},aa2[i-1]);
Aa1:=mtaylor(AA1[i],s = 0,m+1);Aa2:= mtaylor(AA2[i],s = 0,m+1);
TT1:=mtaylor(sa11[i]*Aa1+sa12[i]*Aa2,s=0,m+1):
TT2:=mtaylor(sa21[i]*Aa1+sa22[i]*Aa2,s=0,m+1):
aa1[i]:=simplify(TT1):aa2[i]:=simplify(TT2);
#print('aa[i]',aa1[i]);
od:tete:= aa1[m];quit;
```